APPENDIX B

Equations of Elasticity

B.1 STRAIN-DISPLACEMENT RELATIONS

In general, the concept of *normal strain* is introduced and defined in the context of a uniaxial tension test. The elongated length L of a portion of the test specimen having original length L_0 (the gauge length) is measured and the corresponding normal strain defined as

$$\varepsilon = \frac{L - L_0}{L_0} = \frac{\Delta L}{L_0} \tag{B.1}$$

which is simply interpreted as "change in length per unit original length" and is observed to be a dimensionless quantity. Similarly, the idea of shear strain is often introduced in terms of a simple torsion test of a bar having a circular cross section. In each case, the test geometry and applied loads are designed to produce a simple, uniform state of strain dominated by one major component.

In real structures subjected to routine operating loads, strain is not generally uniform nor limited to a single component. Instead, strain varies throughout the geometry and can be composed of up to six independent components, including both normal and shearing strains. Therefore, we are led to examine the appropriate definitions of strain at a point. For the general case, we denote $u = u(x, y, z)$, $v = v(x, y, z)$, and $w = w(x, y, z)$ as the displacements in the x, y, and z coordinate directions, respectively. (The displacements may also vary with time; for now, we consider only the static case.) Figure B.1(a) depicts an infinitesimal element having undeformed edge lengths $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ located at an arbitrary point (x, y, z) in a solid body. For simplicity, we first assume that this element is loaded in tension in the x direction only and examine the resulting deformation as shown (greatly exaggerated) in Figure B.1(b). Displacement of point P is u while that of point Q is $u + (\partial u/\partial x)\,\mathrm{d}x$ such that the deformed length in the x direction is given by

$$\mathrm{d}x' = \mathrm{d}x + u_Q - u_P = \mathrm{d}x + u + \frac{\partial u}{\partial x}\,\mathrm{d}x - u = \mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}x \tag{B.2}$$

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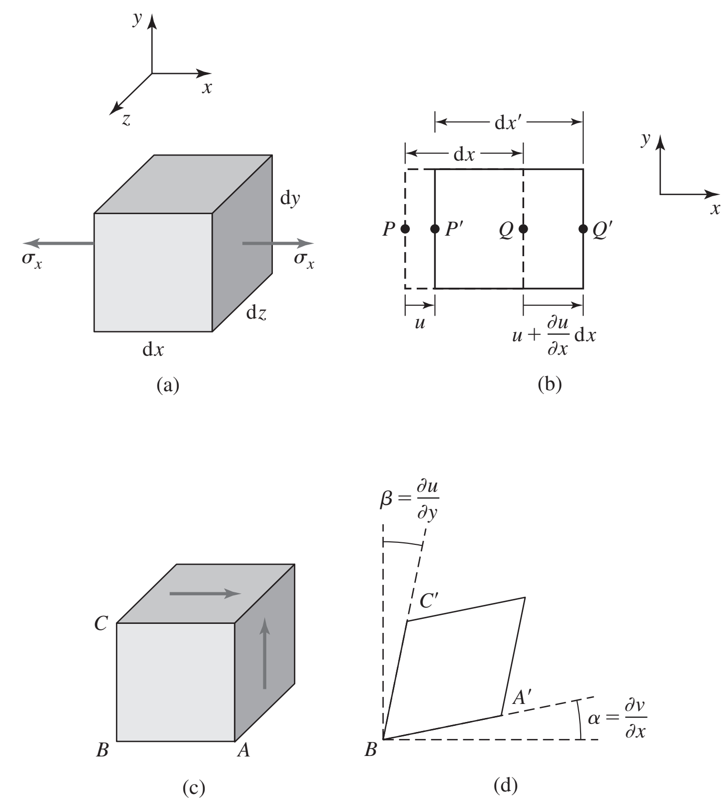

Figure B.1
(a) A differential element in uniaxial stress; (b) resulting axial deformation; (c) differential element subjected to shear; (d) angular changes used to define shear strain.

The normal strain in the x direction at the point depicted is then

$$\varepsilon_x = \frac{dx' - dx}{dx} = \frac{\partial u}{\partial x} \tag{B.3}$$

Similar consideration of changes of length in the y and z directions yields the general definitions of the associated normal strain components as

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \varepsilon_z = \frac{\partial w}{\partial z} \tag{B.4}$$

To examine shearing of the infinitesimal solid, we next consider the situation shown in Figure B.1(c), in which applied surface tractions result in shear of the element, as depicted in Figure B.1(d). Unlike normal strain, the effects of shearing are seen to be distortions of the original rectangular shape of the solid. Such distortion is quantified by angular changes, and we consequently define *shear strain* as a "change in the angle of an angle that was originally a right angle." On first reading, this may sound redundant but it is not. Consider the definition in the context of Figure B.1(c) and B.1(d); angle ABC was a right angle in the undeformed state but has been distorted to $A'BC'$ by shearing. The change of the angle is composed of two parts, denoted α and β, given by the slopes of BA' and BC', respectively as $\partial v/\partial x$ and $\partial u/\partial y$. Thus, the shear strain is

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{B.5}$$

where the double subscript is used to indicate the plane in which the angular change occurs. Similar consideration of distortion in xz and yz planes results in

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \qquad \text{and} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{B.6}$$

as the shear strain components, respectively.

Equations B.3–B.6 provide the basic definitions of the six possible independent strain components in three-dimensional deformation. It must be emphasized that these strain-displacement relations are valid only for small deformations. Additional terms must be included if large deformations occur as a result of geometry or material characteristics. As continually is the case as we proceed, it is convenient to express the strain-displacement relations in matrix form. To accomplish this task, we define the displacement vector as

$$\{\delta\} = \begin{Bmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{Bmatrix} \tag{B.7}$$

(noting that this vector describes a continuous displacement field) and the strain vector as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{B.8}$$

The strain-displacement relations are then expressed in the compact form

$$\{\varepsilon\} = [L]\{\delta\} \tag{B.9}$$

where $[L]$ is the derivative operator matrix given by

$$[L] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \tag{B.10}$$

B.2 STRESS-STRAIN RELATIONS

The equations between stress and strain applicable to a particular material are known as the *constitutive equations* for that material. In the most general type of material possible, it is shown in advanced work in continuum mechanics that the constitutive equations can contain up to 81 independent material constants. However, for a homogeneous, isotropic, linearly elastic material, it is readily shown that only two independent material constants are required to completely specify the relations. These two constants should be quite familiar from elementary strength of materials theory as the modulus of elasticity (Young's modulus) and Poisson's ratio. Again referring to the simple uniaxial tension test, the *modulus of elasticity* is defined as the slope of the stress-strain curve in the elastic region or

$$E = \frac{\sigma_x}{\varepsilon_x} \tag{B.10}$$

where it is assumed that the axis of loading corresponds to the x axis. As strain is dimensionless, the modulus of elasticity has the units of stress usually expressed in lb/in.^{2} or megapascal (MPa).

Poisson's ratio is a measure of the well-known phenomenon that an elastic body strained in one direction also experiences strain in mutually perpendicular directions. In the uniaxial tension test, elongation of the test specimen in the loading direction is accompanied by contraction in the plane perpendicular to the loading direction. If the loading axis is x, this means that the specimen changes dimensions and thus experiences strain in the y and z directions as well, even though no external loading exists in those directions. Formally, Poisson's ratio is defined as

$$\nu = -\frac{\text{unit lateral contraction}}{\text{unit axial elongation}} \tag{B.11}$$

and we note that Poisson's ratio is algebraically positive and the negative sign assures this, since numerator and denominator always have opposite signs. Thus, in the tension test, if ε_x represents the strain resulting from applied load, the induced strain components are given by $\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x$.

The general stress-strain relations for a homogeneous, isotropic, linearly elastic material subjected to a general three-dimensional deformation are as follows:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)] \tag{B.12a}$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_z)] \tag{B.12b}$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)] \tag{B.12c}$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = G\gamma_{xy} \tag{B.12d}$$

$$\tau_{xz} = \frac{E}{2(1+\nu)}\gamma_{xz} = G\gamma_{xz} \tag{B.12e}$$

$$\tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz} = G\gamma_{yz} \tag{B.12f}$$

where we introduce the *shear modulus* or *modulus of rigidity,* defined by

$$G = \frac{E}{2(1+\nu)} \tag{B.13}$$

We may observe from the general relations that the normal components of stress and strain are interrelated in a rather complicated fashion through the Poisson effect but are independent of shear strains. Similarly, the shear stress components* are unaffected by normal strains.

The stress-strain relations can easily be expressed in matrix form by defining the *material property matrix* $[D]$ as

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \tag{B.14}$$

*The double subscript notation used for shearing stresses is explained as follows: The first subscript defines the axial direction perpendicular to the surface on which the shearing stress acts, while the second subscript denotes the axis parallel to the shearing stress. Thus, τ_{xy} denotes a shearing stress acting in the direction of the x axis on a surface perpendicular to the y axis. Via moment equilibrium, it is readily shown that $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, and $\tau_{yz} = \tau_{zy}$.

and writing

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = [D]\{\varepsilon\} = [D][L]\{\delta\} \tag{B.15}$$

Here $\{\sigma\}$ denotes the 6×1 matrix of stress components. We do not use the term *stress vector,* since, as we subsequently observe, that term has a generally accepted meaning quite different from the matrix defined here.

B.3 EQUILIBRIUM EQUATIONS

To obtain the equations of equilibrium for a deformed solid body, we examine the general state of stress at an arbitrary point in the body via an infinitesimal differential element, as shown in Figure B.2. All stress components are assumed to

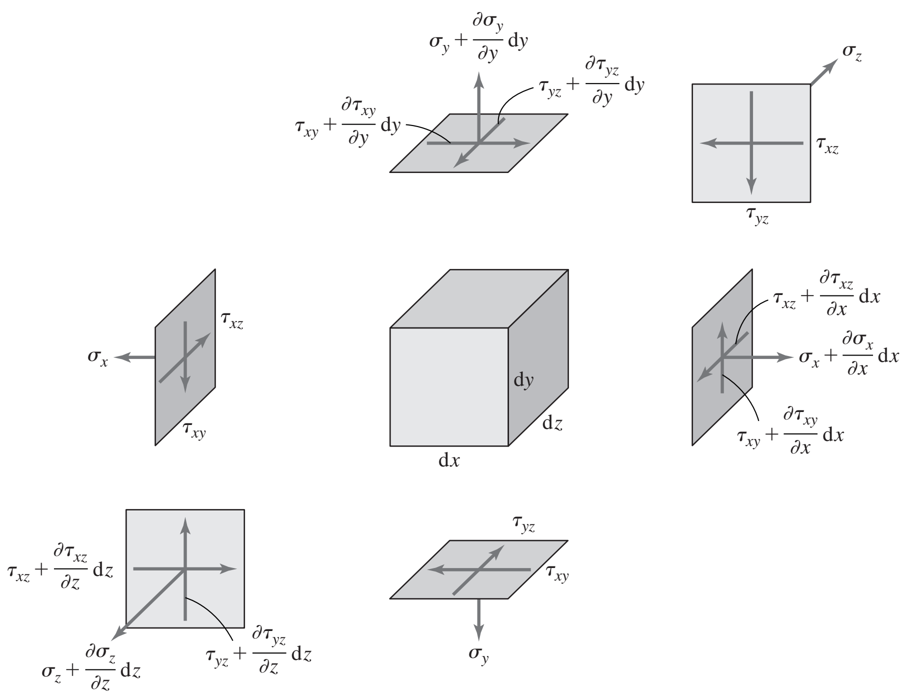

Figure B.2 A three-dimensional element in a general state of stress.

vary spatially, and these variations are expressed in terms of first-order Taylor series expansions, as indicated. In addition to the stress components shown, it is assumed that the element is subjected to a *body force* having axial components B_x, B_y, B_z. The body force is expressed as force per unit volume and represents the action of an external influence that affects the body as a whole. The most common body force is that of gravitational attraction while magnetic and centrifugal forces are also examples.

Applying the condition of force equilibrium in the direction of the x axis for the element of Figure B.2 results in

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y\,\mathrm{d}z - \sigma_x\,\mathrm{d}y\,\mathrm{d}z + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}\,\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}z - \tau_{xy}\,\mathrm{d}x\,\mathrm{d}z$$
$$+ \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z}\,\mathrm{d}z\right)\mathrm{d}x\,\mathrm{d}y - \tau_{xz}\,\mathrm{d}x\,\mathrm{d}y + B_x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0 \tag{B.16}$$

Expanding and simplifying Equation B.16 yields

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0 \tag{B.17}$$

Similarly, applying the force equilibrium conditions in the y and z coordinate directions yields

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + B_y = 0 \tag{B.18}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + B_z = 0 \tag{B.19}$$

respectively.

B.4 COMPATIBILITY EQUATIONS

Equations B.3–B.6 define six strain components in terms of three displacement components. A fundamental premise of the theory of continuum mechanics is that a continuous body remains continuous during and after deformation. Therefore, the displacement and strain functions must be continuous and single valued. Given a continuous displacement field u, v, w, it is straightforward to compute continuous, single-valued strain components via the strain-displacement relations. However, the inverse case is a bit more complicated. That is, given a field of six continuous, single-valued strain components, we have six partial differential equations to solve to obtain the displacement components. In this case, there is no assurance that the resulting displacements will meet the requirements of continuity and single-valuedness. To ensure that displacements are continuous when computed in this manner, additional relations among the strain components have been derived, and these are known as the *compatibility equations*. There are six independent compatibility equations, one of which is

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{B.20}$$

The other five equations are similarly second-order relations. While not used explicitly in this text, the compatibility equations are absolutely essential in advanced methods in continuum mechanics and the theory of elasticity.